

ENUMERATION OF LINEAR TRANSFORMATION SHIFT REGISTERS

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ABSTRACT. We consider the problem of counting the number of linear transformation shift registers of a given order over a finite field. The problem is equivalent to the enumeration of a certain subset of block companion matrices. We derive some explicit formulae and use them to deduce a theorem of Carlitz on the number of self-reciprocal irreducible polynomials of a given degree over a finite field.

1. INTRODUCTION

Let m, n be positive integers, q be a prime power and $\alpha \in \mathbb{F}_{q^{mn}}$. An m -dimensional \mathbb{F}_q -linear subspace of $\mathbb{F}_{q^{mn}}$ is said to be α -splitting if

$$\mathbb{F}_{q^{mn}} = W \oplus \alpha W \oplus \cdots \oplus \alpha^{n-1}W.$$

Splitting subspaces were studied by Niederreiter [13] in the context of his work on the multiple recursive matrix method for pseudorandom number generation. In his paper [13, p. 11], he asked the following question which was stated as an open problem: If α generates the cyclic group $\mathbb{F}_{q^{mn}}^*$, what is the number of m -dimensional α -splitting subspaces? More generally, we may ask

Question 1.1. *Given $\alpha \in \mathbb{F}_{q^{mn}}$ such that $\mathbb{F}_{q^{mn}} = \mathbb{F}_q(\alpha)$, what is the number of m -dimensional α -splitting subspaces?*

We refer to Ghorpade, Hasan and Kumari [7], Ghorpade and Ram [5, 6] and Chen and Tseng [2] for recent progress on the above question. In particular, the work of Chen and Tseng settles Question (1.1) completely by proving a conjecture of Ghorpade and Ram [5, Conj. 5.5]. A related conjecture arose in the study of word-oriented linear feedback shift registers (σ -LFSRs) by Zeng, Han and He [17] in connection with their work on stream ciphers. It was shown in [5] that their notion of σ -LFSR is essentially equivalent to that of a splitting subspace defined by Niederreiter. More precisely, the problem (1.1) of enumeration of splitting subspaces was shown to be equivalent to counting a subclass of block companion matrices which turn out to be the state transition matrices of certain σ -LFSRs.

A subcategory of σ -LFSRs called transformation shift registers (TSRs) was considered by Tsaban and Vishne [16] in order to solve a problem of Preneel [14]. It turns out that the TSRs have very good cryptographic properties, especially when the corresponding characteristic polynomial is irreducible or primitive.

We define a (m, n) -TSR matrix over \mathbb{F}_q to be a matrix $T \in M_{mn}(\mathbb{F}_q)$ of the form

$$T = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdot & \cdot & \mathbf{0} & \mathbf{0} & A \\ I_m & \mathbf{0} & \mathbf{0} & \cdot & \cdot & \mathbf{0} & \mathbf{0} & c_1 A \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdot & \cdot & I_m & \mathbf{0} & c_{n-2} A \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdot & \cdot & \mathbf{0} & I_m & c_{n-1} A \end{pmatrix},$$

where $c_1, \dots, c_{n-1} \in \mathbb{F}_q$, $A \in M_m(\mathbb{F}_q)$ and I_m denotes the $m \times m$ identity matrix over \mathbb{F}_q , while $\mathbf{0}$ indicates the zero matrix in $M_m(\mathbb{F}_q)$. We denote by $TSR(m, n; q)$ the set of all (m, n) -TSR matrices over \mathbb{F}_q . It can be shown [8, p. 5] that matrices in $TSR(m, n; q)$ are precisely the state transition matrices¹ of transformation shift registers of order n over \mathbb{F}_{q^m} .

While σ -LFSRs have been studied in great detail, very little is known about the TSRs; indeed, given positive integers m, n and a prime power q , it is not even known if there exists a matrix in $TSR(m, n; q)$ with an irreducible characteristic polynomial.

In this article, we adopt a matrix theoretic approach to enumerating TSRs by considering their state transition matrices. Thus, we are mainly interested in $|TSR(m, n; q)|$ and the number of elements in $TSR(m, n; q)$ which have an irreducible characteristic polynomial. A formula (5.4) for the number of matrices in $TSR(m, 2; q)$ with an irreducible characteristic polynomial is obtained. We then give a simple method to construct certain TSRs from self-reciprocal polynomials. Finally, we use the results on TSR matrices to deduce a theorem of Carlitz [1] (which has been proved by Cohen [3], Meyn [10], Meyn and Götz [11] and Miller [12] in a variety of ways) on the number of self-reciprocal irreducible monic polynomials of a given degree.

2. PRELIMINARIES

The map which associates to a $mn \times mn$ matrix its characteristic polynomial, viz.,

$$\Phi : M_{mn}(\mathbb{F}_q) \rightarrow \mathbb{F}_q[X] \quad \text{defined by} \quad \Phi(T) := \det(XI_{mn} - T)$$

will often be referred to as the *characteristic map*. The restriction of Φ to (m, n) -TSR matrices will be denoted by $\Phi_{(m, n)}$. We denote by $TSRP(m, n; q)$ the set of (m, n) -TSR matrices which have a primitive characteristic polynomial over \mathbb{F}_q , and by $TSRI(m, n; q)$ the set of (m, n) -TSR matrices over \mathbb{F}_q that have an irreducible characteristic polynomial. For each positive integer r , we denote by $\mathcal{J}(r; q)$ and $\mathcal{P}(r; q)$ the set of irreducible polynomials of degree r and the set of primitive polynomials of degree r respectively. Thus Φ maps $TSRI(m, n; q)$ into

¹For reasons that will become clear, our definition of state transition matrix of a TSR differs slightly from the one in [8, p. 5]. Note, however, that the two definitions coincide in the case of TSR matrices which are nonsingular.

$\mathcal{J}(mn; q)$ and $TSRP(m, n; q)$ into $\mathcal{P}(mn; q)$. As a result, restrictions of Φ yield the following maps:

$$(1) \quad \Phi_P : TSRP(m, n; q) \rightarrow \mathcal{P}(mn; q) \quad \text{and} \quad \Phi_I : TSRI(m, n; q) \rightarrow \mathcal{J}(mn; q).$$

It is easy to show [8, Lem. 4.2] that if $T \in TSR(m, n; q)$, then

$$(2) \quad \Phi(T) = \det(X^n I_n - g_T(X) T_{(m)}),$$

where $g_T(X) = 1 + c_1 X + \dots + c_{n-1} X^{n-1} \in \mathbb{F}_q[X]$ and $T_{(m)}$ denotes the submatrix of T formed by the first m rows and last m columns of T . Note that T is uniquely determined by $g_T(X)$ and $T_{(m)}$. For every matrix M we denote by $\phi_M(X)$ the characteristic polynomial of M . It follows easily from (2) that

$$(3) \quad \phi_T(X) = g_T(X)^m \phi_{T_{(m)}}\left(\frac{X^n}{g_T(X)}\right)$$

Thus if $\phi_T(X)$ is irreducible in $\mathbb{F}_q[X]$, then so is $\phi_{T_{(m)}}(X)$. However, the converse is not true in general. For example if $g_T(X) = 1$, then

$$\phi_T(X) = \phi_{T_{(m)}}(X^n)$$

which is not irreducible when n is a multiple of q . If $\phi_T(X)$ is primitive in $\mathbb{F}_q[X]$, then it is not necessarily true that $\phi_{T_{(m)}}(X)$ is primitive. Consider $T \in TSR(1, 2; 3)$ given by

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

In this case $\phi_T(X) = X^2 - X - 1$ is primitive but $\phi_{T_{(m)}}(X) = X - 1$ is not.

The next proposition describes the form of $\phi_{T_{(m)}}(X)$ when $T \in TSRP(m, n; q)$. First, we need a lemma.

Lemma 2.1. *If N is a positive integer and $f(X) \in \mathcal{P}(N; q)$ then $(-1)^N f(0)$ is a primitive element of \mathbb{F}_q .*

Proof. See [9, Thm. 3.18]. □

Proposition 2.2. *If $T \in TSRP(m, n; q)$ then*

$$(-1)^{m(n+1)} \phi_{T_{(m)}}((-1)^{n+1} X) \in \mathcal{P}(m; q).$$

Proof. Let $\alpha_i (1 \leq i \leq m)$ be the roots of $\phi_{T_{(m)}}(X)$ (which is necessarily irreducible in $\mathbb{F}_q[X]$) in \mathbb{F}_{q^m} . Then

$$\phi_T(X) = \prod_{i=1}^m (X^n - \alpha_i g_T(X))$$

is a factorization of $\phi_T(X)$ into irreducible polynomials in $\mathbb{F}_{q^m}[X]$. Then, for each i , $X^n - \alpha_i g_T(X)$ is necessarily primitive in $\mathbb{F}_{q^m}[X]$. By Lemma 2.1, $(-1)^{n+1} \alpha_i$ is

primitive in \mathbb{F}_{q^m} for each i . It is easily seen that the m elements $(-1)^{n+1}\alpha_i$ are also conjugates of each other over \mathbb{F}_q . Equivalently,

$$\prod_{i=1}^m (X + (-1)^n \alpha_i) \in \mathcal{P}(m; q).$$

This is equivalent to the statement of the proposition. \square

Corollary 2.3. *If $\text{char}(\mathbb{F}_q) = 2$ and $\phi_T(X)$ is primitive, then so is $\phi_{T(m)}(X)$.*

Corollary 2.4. *If n is odd and $\phi_T(X)$ is primitive, then so is $\phi_{T(m)}(X)$.*

3. FIBERS OF THE CHARACTERISTIC MAP

The maps Φ_I and Φ_P defined in (1) are not surjective in general. To see this, let $T \in \text{TSR}(2, 2; 2)$. We show that the primitive polynomial $X^4 + X + 1 \in \mathbb{F}_2[X]$ cannot be the characteristic polynomial of T . Suppose, to the contrary, that

$$\phi_T(X) = X^4 + X + 1.$$

Let $\phi_{T(m)}(X) = X^2 + aX + b$. Then

$$X^4 + aX^2 g_T(X) + b g_T(X)^2 = X^4 + X + 1.$$

Formally differentiating with respect to X on both sides, we obtain

$$aX^2 g_T'(X) = 1$$

which is impossible.

Since Φ_I is not surjective in general, the following natural question arises.

Question 3.1. *Which polynomials $f(X) \in \mathbb{F}_q[X]$ are the characteristic polynomial of some $T \in \text{TSRI}(m, n; q)$ and what is the cardinality of the fiber $\Phi_{(m,n)}^{-1}(f(X))$.*

It follows from (3) that if $f(X) \in \Phi(\text{TSR}(m, n; q))$ then $f(X)$ is necessarily of the form

$$(4) \quad g(X)^m h\left(\frac{X^n}{g(X)}\right)$$

for some monic polynomial $h(X) \in \mathbb{F}_q[X]$ of degree m and a not necessarily monic $g(X) \in \mathbb{F}_q[X]$ of degree at most $n - 1$ with $g(0) = 1$.

We say that a polynomial $f(X) \in \mathbb{F}_q[X]$ is (m, n) -decomposable if it is the characteristic polynomial of some (m, n) -TSR matrix. We refer to (4) as an (m, n) -decomposition of $f(X)$. We further say that $f(X)$ is *uniquely* (m, n) -decomposable if the representation of f in the form (4) is unique.

The following theorem will be used to provide a partial answer to Question 3.1.

Theorem 3.2. *Let $f(X) \in \mathbb{F}_q[X]$ be a monic polynomial of degree n and let $f = f_1^{m_1} \cdots f_k^{m_k}$, where the f_i are distinct irreducible polynomials in $\mathbb{F}_q[X]$ of degree d_i . The number of matrices in $M_n(\mathbb{F}_q)$ that have $f(X)$ as their characteristic polynomial is given by*

$$N_\chi(f(X)) = q^{n^2-n} \frac{F(q, n)}{\prod_{i=1}^k F(q^{d_i}, m_i)}.$$

where

$$F(q, r) = \prod_{i=1}^r (1 - q^{-i})$$

for every positive integer r .

Proof. See [4, §2] or [15, Thm. 2]. □

Theorem 3.3. *Suppose $f(X)$ is uniquely (m, n) -decomposable as*

$$g(X)^m h\left(\frac{X^n}{g(X)}\right).$$

Then,

$$|\Phi_{(m,n)}^{-1}(f(X))| = N_\chi(h(X)).$$

Proof. Suppose $T \in TSR(m, n; q)$ and $\phi_T(X) = f(X)$. By the hypothesis, $g_T(X)$ and $\phi_{T_{(m)}}(X)$ are uniquely determined and are equal to $g(X)$ and $h(X)$ respectively. Thus the number of such T is equal to the number of possible values of $T_{(m)}$ with $\phi_{T_{(m)}}(X) = h(X)$. This is the statement of the theorem. □

Corollary 3.4. *Suppose $T \in TSR(m, n; q)$ is such that $\phi_T(X)$ is uniquely (m, n) -decomposable. Then*

$$|\Phi_{(m,n)}^{-1}(\phi_T(X))| = N_\chi(\phi_{T_{(m)}}(X)).$$

Proposition 3.5. *If $f(X) = X^{mn}$ then*

$$|\Phi_{(m,n)}^{-1}(f(X))| = (q^{m^2-m} - 1) q^{n-1} + 1.$$

Proof. In this case $\Phi_{(m,n)}^{-1}(f(X))$ is precisely the set of those $T \in TSR(m, n; q)$ for which $T_{(m)}$ is nilpotent. Thus there are $N_\chi(X^m)$ possible values for $T_{(m)}$. There is precisely one T for which $T_{(m)} = \mathbf{0}$. For $T_{(m)} \neq \mathbf{0}$, $g_T(X)$ can be arbitrarily chosen and the number of such T is $(N_\chi(X^m) - 1) q^{n-1}$. Adding the two values, we obtain

$$\begin{aligned} |\Phi_{(m,n)}^{-1}(X^{mn})| &= (N_\chi(X^m) - 1) q^{n-1} + 1 \\ &= (q^{m^2-m} - 1) q^{n-1} + 1. \end{aligned}$$

□

Theorem 3.6. Suppose $f(X)$ is (m, n) -decomposable and $f(X) \neq X^{mn}$. Let

$$f(X) = g_{ij}(X)^m h_i \left(\frac{X^n}{g_{ij}(X)} \right) \quad 1 \leq i \leq r, \quad 1 \leq j \leq n_i$$

be all possible (m, n) -decompositions of f , where the pairs (h_i, g_{ij}) are distinct for $1 \leq i \leq r$ and $1 \leq j \leq n_i$. Then,

$$\left| \Phi_{(m,n)}^{-1}(f(X)) \right| = \sum_{i=1}^r n_i N_\chi(h_i(X)).$$

Proof. For each pair (h_i, g_{ij}) with $1 \leq i \leq r, 1 \leq j \leq n_i$, the number of $T \in TSR(m, n; q)$ with $g_T = g_{ij}$ and $\phi_{T(m)} = h_i$ is $N_\chi(h_i)$. Since $f(X) \neq X^{mn}$, $T_{(m)}$ is necessarily nonzero and thus no T is counted more than once. Summing over i, j , we obtain

$$\begin{aligned} \left| \Phi_{(m,n)}^{-1}(f(X)) \right| &= \sum_{i=1}^r \sum_{j=1}^{n_i} N_\chi(h_i(X)) \\ &= \sum_{i=1}^r n_i N_\chi(h_i(X)). \end{aligned}$$

□

Theorem 3.7. Suppose $f(X) \in \mathbb{F}_q[X]$ is (m, n) -decomposable with

$$f(X) = g(X)^m h \left(\frac{X^n}{g(X)} \right)$$

and

$$h(X) = X^m + \sum_{i=1}^m a_i X^{m-i}.$$

Suppose further that not all a_i are zero and that $a_i = 0$ whenever $\text{char}(\mathbb{F}_q) \mid i$. Then $f(X)$ is uniquely (m, n) -decomposable.

Proof. Suppose

$$(5) \quad f(X) = g_1(X)^m h_1 \left(\frac{X^n}{g_1(X)} \right) = g_2(X)^m h_2 \left(\frac{X^n}{g_2(X)} \right)$$

with

$$h_1(X) = X^m + \sum_{i=1}^m a_i X^{m-i} \quad \text{and} \quad h_2(X) = X^m + \sum_{i=1}^m b_i X^{m-i}.$$

Setting $X = 0$ we see that $a_m = b_m$. If $a_m = b_m = 0$, then by a similar argument, $a_{m-1} = b_{m-1}$. Let $k = \max\{r : a_r \neq 0\}$ (k exists since not all a_i are zero). Then,

by the above reasoning, we must have $k = \max\{r : b_r \neq 0\}$ and $a_k = b_k$. By the hypothesis $p = \text{char}(\mathbb{F}_q)$ does not divide k . From (5), we obtain

$$X^{mn} + \sum_{i=1}^k a_i X^{n(m-i)} g_1(X)^i = X^{mn} + \sum_{i=1}^k b_i X^{n(m-i)} g_2(X)^i$$

and some rearrangement shows that

$$X^n \mid a_k (g_2(X)^k - g_1(X)^k).$$

Since $p \nmid k$ the k -th roots of unity in \mathbb{F}_q , say w_1, \dots, w_k , are distinct. Thus, X^n divides

$$\prod_{i=1}^k (g_2(X) - w_i g_1(X)).$$

Since $g_1(0) = g_2(0) = 1$, the only factor in the above product with a nonzero constant term is $g_2(X) - g_1(X)$ which is a polynomial of degree at most $n - 1$. Then X^n divides $g_2(X) - g_1(X)$ and this is possible only if $g_1(X) = g_2(X)$. From this it easily follows that $h_1(X) = h_2(X)$ and the theorem follows. \square

Corollary 3.8. *Suppose $m < \text{char}(\mathbb{F}_q)$ and $T \in \text{TSR}(m, n; q)$ is such that $\phi_T(X) \neq X^{mn}$. Then, $\phi_T(X)$ is uniquely (m, n) -decomposable.*

Proof. Using the notation of the theorem, $f(X) = \phi_T(X)$ and $h(X) = \phi_{T(m)}(X)$. Since $\phi_T(X) \neq X^{mn}$ it follows that $\phi_{T(m)}(X) \neq X^m$. The hypothesis that $m < \text{char}(\mathbb{F}_q)$ ensures that $h(X)$ satisfies the hypothesis of the above theorem and the corollary follows. \square

Theorem 3.9. *Suppose $f(X)$ is (m, n) -decomposable and irreducible in $\mathbb{F}_q[X]$. Then $f(X)$ is uniquely (m, n) -decomposable.*

Proof. Let

$$f(X) = g_1(X)^m h_1\left(\frac{X^n}{g_1(X)}\right) = g_2(X)^m h_2\left(\frac{X^n}{g_2(X)}\right)$$

be two (m, n) -decompositions of $f(X)$. Since f is irreducible, so are h_1 and h_2 . Let

$$h_1(X) = \prod_{i=1}^m (X - \lambda_i) \quad \text{and} \quad h_2(X) = \prod_{i=1}^m (X - \mu_i)$$

be the factorizations of h_1 and h_2 in $\mathbb{F}_{q^m}[X]$. Then

$$\prod_{i=1}^m (X^n - \lambda_i g_1(X)) \quad \text{and} \quad \prod_{i=1}^m (X^n - \mu_i g_2(X))$$

are two factorizations of $f(X)$ in $\mathbb{F}_{q^m}[X]$. Since $f(X)$ is irreducible of degree mn , $f(X)$ splits uniquely into m distinct irreducible factors of degree n in $\mathbb{F}_{q^m}[X]$. Thus each factor in both the above products is irreducible and the factors in one

product are merely a rearrangement of those in the other. Thus there exists a permutation $\sigma \in \mathfrak{S}_m$ such that

$$X^n - \lambda_i g_1(X) = X^n - \mu_{\sigma(i)} g_2(X) \quad \text{for } 1 \leq i \leq m.$$

Since $g_1(0) = g_2(0)$, it follows that $\lambda_i = \mu_{\sigma(i)}$ for $1 \leq i \leq m$ and hence $g_1(X) = g_2(X)$. Since the λ_i are a permutation of the μ_j it follows that $h_1(X) = h_2(X)$ as well, proving uniqueness. \square

Theorem 3.10. *If $T \in TSRI(m, n; q)$ then*

$$\left| \Phi_{(m,n)}^{-1}(\phi_T(X)) \right| = \frac{|GL_m(\mathbb{F}_q)|}{q^m - 1}.$$

Proof. If T is as above then $\phi_T(X)$ is irreducible and (m, n) -decomposable. By Theorem 3.9 $\phi_T(X)$ is uniquely (m, n) -decomposable, and by Corollary 3.4

$$\left| \Phi_{(m,n)}^{-1}(\phi_T(X)) \right| = N_\chi(\phi_{T(m)}(X)).$$

Since $\phi_{T(m)}(X)$ is also irreducible it follows from Theorem 3.2 that

$$N_\chi(\phi_{T(m)}(X)) = \frac{|GL_m(\mathbb{F}_q)|}{q^m - 1}.$$

\square

4. TSR MATRICES WITH AN IRREDUCIBLE CHARACTERISTIC POLYNOMIAL

We now restrict our attention to TSR matrices with an irreducible characteristic polynomial.

Theorem 4.1.

$$\begin{aligned} |TSRI(1, n; q)| &= |\mathcal{I}(n; q)| \\ |TSRI(m, 1; q)| &= \frac{|GL_m(\mathbb{F}_q)|}{q^m - 1} |\mathcal{I}(m; q)| \\ |TSRP(1, n; q)| &= |\mathcal{P}(n; q)| \\ |TSRP(m, 1; q)| &= \frac{|GL_m(\mathbb{F}_q)|}{q^m - 1} |\mathcal{P}(m; q)| \end{aligned}$$

Proof. If either m or n equals 1, it is easily seen that the maps Φ_I and Φ_P are surjective. The above formulae follow easily from Theorem 3.10. \square

Let $S_q(m, n)$ denote the set of irreducible polynomials $f(X) \in \mathbb{F}_{q^m}[X]$ of the form

$$X^n - \lambda g(X)$$

where λ satisfies $\mathbb{F}_{q^m} = \mathbb{F}_q(\lambda)$ and $g(X) \in \mathbb{F}_q[X]$ with $g(0) = 1$ and $\deg g(X) \leq n - 1$. The significance of $S_q(m, n)$ is apparent from the following theorem.

Theorem 4.2. *For positive integers m, n*

$$|TSRI(m, n; q)| = \frac{|S_q(m, n)|}{m} \frac{|GL_m(\mathbb{F}_q)|}{q^m - 1}.$$

Proof. Define

$$\Delta_q(m, n) := \Phi_I(TSRI(m, n; q)).$$

By Theorem 3.10

$$(6) \quad |TSRI(m, n; q)| = |\Delta_q(m, n)| \frac{|GL_m(\mathbb{F}_q)|}{q^m - 1}.$$

Define a map

$$\Gamma : S_q(m, n) \rightarrow \mathbb{F}_{q^{mn}}[X]$$

by

$$\Gamma((X^n - \lambda g(X))) := \prod_{i=0}^{m-1} (X^n - \lambda^{q^i} g(X)).$$

It is easy to see that the product on the right is (m, n) -decomposable. Let β be a root of $X^n - \lambda g(X)$ in some extension field of \mathbb{F}_{q^m} . Then, the minimal polynomial of β over \mathbb{F}_q is clearly $\Gamma(X^n - \lambda g(X))$. Thus $\Gamma(X^n - \lambda g(X))$ is irreducible in $\mathbb{F}_q[X]$. Since $\Delta_q(m, n)$ is precisely the set of irreducible (m, n) -decomposable polynomials in $\mathbb{F}_q[X]$, it follows that $\Gamma(S_q(m, n)) \subseteq \Delta_q(m, n)$. We claim that

$$\Gamma(S_q(m, n)) = \Delta_q(m, n).$$

To see this, let $f(X) \in \Delta_q(m, n)$. Since f is irreducible, f has a unique (m, n) -decomposition, say

$$f(X) = g(X)^m h\left(\frac{X^n}{g(X)}\right).$$

Then $h(X)$ is necessarily irreducible in $\mathbb{F}_q[X]$ and if μ is a root of $h(X)$ in \mathbb{F}_{q^m} , then

$$\Gamma(X^n - \mu g(X)) = f(X),$$

proving the claim. It is now easy to see that $\Gamma^{-1}(f(X))$ is precisely the set $\{X^n - \mu^{q^i} g(X) : 0 \leq i \leq m-1\}$. Thus $|\Gamma^{-1}(f)| = m$ for each $f \in \Delta_q(m, n)$ and consequently

$$|\Delta_q(m, n)| = \frac{|S_q(m, n)|}{m}.$$

The theorem now follows from (6). □

5. THE CASE $n = 2$

In what follows, we denote $|S_q(m, n)|$ by $N_q(m, n)$. We now consider the computation of $|TSRI(m, 2; q)|$ for $m > 1$. In this case,

$$\begin{aligned} N_q(m, 2) &= |\{X^2 - \lambda(aX + 1) \text{ irreducible in } \mathbb{F}_{q^m}[X] : \mathbb{F}_{q^m} = \mathbb{F}_q(\lambda), a \in \mathbb{F}_q\}| \\ &= |\{X^2 + aX - \alpha \text{ irreducible in } \mathbb{F}_{q^m}[X] : \mathbb{F}_{q^m} = \mathbb{F}_q(\alpha), a \in \mathbb{F}_q\}|. \end{aligned}$$

For every positive integer $t > 1$ and $a \in \mathbb{F}_q$, define

$$(7) \quad V_t(a) = \{\alpha \in \mathbb{F}_{q^t} : \mathbb{F}_{q^t} = \mathbb{F}_q(\alpha), X^2 + aX - \alpha \text{ is irreducible in } \mathbb{F}_{q^t}[X]\}.$$

Then it follows that

$$N_q(m, 2) = \sum_{a \in \mathbb{F}_q} |V_m(a)|.$$

Proposition 5.1. *For $m > 1$ and $a \in \mathbb{F}_q$, $V_m(a) = \emptyset$ if and only if q is even and $a = 0$.*

Proof. Define

$$Z_m := \{\alpha \in \mathbb{F}_{q^m} : \mathbb{F}_{q^m} = \mathbb{F}_q(\alpha)\}.$$

If q is even and $a = 0$ then

$$\begin{aligned} V_m(a) &= \{\alpha \in \mathbb{F}_{q^m} : \mathbb{F}_{q^m} = \mathbb{F}_q(\alpha), X^2 - \alpha \text{ is irreducible in } \mathbb{F}_{q^m}[X]\} \\ &= \emptyset. \end{aligned}$$

since every element in \mathbb{F}_{q^m} is a square. Now suppose either q is odd or $a \neq 0$. If the polynomial $X^2 + aX - \alpha$ is reducible for some $\alpha \in Z_m$, then it has two roots, say β and $-a - \beta$ (which are distinct by the assumptions on q, a and m), that necessarily lie in Z_m . Thus

$$Z_m \not\subseteq \{x^2 + ax : x \in Z_m\}.$$

Now observe that if $x^2 + ax \in Z_m$ for some $x \in \mathbb{F}_{q^m}$ then $x \in Z_m$. Thus

$$Z_m \not\subseteq \{x^2 + ax : x \in \mathbb{F}_{q^m}\}$$

which implies that $V_m(a)$ is nonempty. \square

We will use the above proposition implicitly in the proof of the next theorem.

Theorem 5.2. *Suppose $m > 1$ and $m = 2^k l$ where k, l are nonnegative integers with l odd.*

(1) *If $l = 1$, then*

$$N_q(m, 2) = \begin{cases} \frac{(q-1)q^m}{2} & q \text{ even,} \\ \frac{q(q^m-1)}{2} & q \text{ odd.} \end{cases}$$

(2) If $l > 1$, then

$$N_q(m, 2) = \begin{cases} \frac{l|\mathcal{J}(l; q^{2^k})|}{2}(q-1) & q \text{ even}, \\ \frac{l|\mathcal{J}(l; q^{2^k})|}{2}q & q \text{ odd}. \end{cases}$$

Proof. For each positive integer $t > 1$, let

$$Z_t = \{\alpha \in \mathbb{F}_{q^t} : \mathbb{F}_{q^t} = \mathbb{F}_q(\alpha)\}$$

as in Proposition 5.1. Let $a \in \mathbb{F}_q$ and assume that $a \neq 0$ whenever q is even. Define for each positive integer $t > 1$ the sets

$$\begin{aligned} X_t(a) &= \{\alpha \in \mathbb{F}_{q^t} : \mathbb{F}_{q^t} = \mathbb{F}_q(\alpha^2 + a\alpha)\}, \\ Y_t(a) &= \{\alpha \in \mathbb{F}_{q^t} : \mathbb{F}_{q^t} = \mathbb{F}_q(\alpha) \neq \mathbb{F}_q(\alpha^2 + a\alpha)\}, \\ U_t(a) &= \{\alpha^2 + a\alpha : \alpha \in \mathbb{F}_{q^t}, \mathbb{F}_{q^t} = \mathbb{F}_q(\alpha^2 + a\alpha)\}. \end{aligned}$$

If $V_t(a)$ is as in (7), then it is easy to see that

$$(8) \quad Z_t = X_t(a) \sqcup Y_t(a) = U_t(a) \sqcup V_t(a).$$

Denote the cardinalities of $Z_t, X_t(a), Y_t(a), U_t(a), V_t(a)$ by z_t, x_t, y_t, u_t, v_t respectively. Then by (8), it follows that $z_t = x_t + y_t = u_t + v_t$. For each $t > 1$, the function $h(x) = x^2 + ax$ maps $X_t(a)$ onto $U_t(a)$ and $Y_{2t}(a)$ onto $V_t(a)$. Thus

$$x_t = 2u_t \quad \text{and} \quad y_{2t} = 2v_t \quad (t > 1).$$

For $0 \leq i \leq k$ let $m_i = m/2^i$. Then, for $1 \leq i \leq k$ and $m_{i-1} > 1$, we have

$$x_{m_i} = 2u_{m_i} \quad \text{and} \quad y_{m_i} = 2v_{m_{i-1}}.$$

If m is odd, then $m \geq 3$ and $y_m = 0$ since a field extension of odd degree cannot contain any extension of degree 2. If m is even, then

$$\begin{aligned} y_m + x_{m_1} &= 2(v_{m_1} + u_{m_1}) \\ &= 2(x_{m_1} + y_{m_1}). \end{aligned}$$

Thus

$$\begin{aligned} y_m &= 2(v_{m_1} + u_{m_1}) - x_{m_1} \\ &= z_{m_1} + y_{m_1}. \end{aligned}$$

The solution to the recurrence depends on m . If m is a power of 2 ($m = 2^k$), then

$$\begin{aligned} (9) \quad y_m &= y_{m_{k-1}} + \sum_{i=1}^{k-1} z_{m_i} \\ &= y_2 + \sum_{i=1}^{k-1} z_{m_i} \end{aligned}$$

where the second summand is understood to be zero when $k = 1$. If m is not a power of 2 (i.e. $l > 1$), then

$$(10) \quad \begin{aligned} y_m &= y_{m_k} + \sum_{i=1}^k z_{m_i} \\ &= \sum_{i=1}^k z_{m_i} \end{aligned}$$

since $m_k = l(\geq 3)$ is odd.

It now remains to compute (9) and (10). First consider (9) (where $m = 2^k$). If r is a power of 2, then $z_r = q^r - q^{r/2}$. A simple calculation shows that

$$y_m = q^{m/2} - q + y_2.$$

Now

$$\begin{aligned} y_2 &= |\{\alpha \in \mathbb{F}_{q^2} : \mathbb{F}_{q^2} = \mathbb{F}_q(\alpha) \neq \mathbb{F}_q(\alpha^2 + a\alpha)\}| \\ &= 2 |\{\alpha \in \mathbb{F}_q : X^2 + aX - \alpha \text{ is irreducible in } \mathbb{F}_q[X]\}| \\ &= 2 (q - |\{s^2 + as : s \in \mathbb{F}_q\}|) \\ &= \begin{cases} q-1 & q \text{ odd,} \\ q & q \text{ even.} \end{cases} \end{aligned}$$

Therefore

$$\begin{aligned} |V_m(a)| &= v_m = z_m - u_m \\ &= \frac{z_m + y_m}{2} \\ &= \frac{q^m - q + y_2}{2}. \end{aligned}$$

Thus

$$\begin{aligned} N_q(m, 2) &= \sum_{a \in \mathbb{F}_q} |V_m(a)| = \begin{cases} |V_m(1)|(q-1) & q \text{ even,} \\ |V_m(1)|q & q \text{ odd.} \end{cases} \\ &= \begin{cases} \frac{(q-1)q^m}{2} & q \text{ even,} \\ \frac{q(q^m-1)}{2} & q \text{ odd.} \end{cases} \end{aligned}$$

This settles the first part of the theorem. In the second case ($m = 2^k l$, $l > 1$), we have

$$\begin{aligned}
 v_m = \frac{z_m + y_m}{2} &= \frac{1}{2} \sum_{i=0}^k z_{m_i} && \text{from (10)} \\
 &= \frac{1}{2} \sum_{i=0}^k z_{2^i l} \\
 &= \frac{z_l}{2} + \frac{1}{2} \sum_{i=1}^k z_{2^i l} \\
 &= \frac{z_l}{2} + \frac{1}{2} \sum_{i=1}^k \sum_{d|2^i l} \mu(d) q^{\frac{2^i l}{d}} \\
 &= \frac{z_l}{2} + \frac{1}{2} \sum_{i=1}^k \sum_{d|2l} \mu(d) q^{\frac{2^i l}{d}}
 \end{aligned}$$

since $\mu(d) = 0$ if $4 \mid d$. Since $\mu(2d) = -\mu(d)$ for odd d we can rewrite this as

$$\begin{aligned}
 &\frac{z_l}{2} + \frac{1}{2} \sum_{i=1}^k \sum_{d|l} \mu(d) \left(q^{\frac{2^i l}{d}} - q^{\frac{2^{i-1} l}{d}} \right) \\
 &= \frac{z_l}{2} + \frac{1}{2} \sum_{d|l} \mu(d) \left(q^{\frac{2^k l}{d}} - q^{\frac{l}{d}} \right) \\
 &= \frac{z_l}{2} + \left(\frac{1}{2} \sum_{d|l} \mu(d) q^{\frac{2^k l}{d}} \right) - \frac{z_l}{2} \\
 &= \frac{l|\mathcal{J}(l; q^{2^k})|}{2}.
 \end{aligned}$$

Thus

$$N_q(m, 2) = \sum_{a \in \mathbb{F}_q} |V_m(a)| = \begin{cases} \frac{l|\mathcal{J}(l; q^{2^k})|}{2} (q-1) & q \text{ even,} \\ \frac{l|\mathcal{J}(l; q^{2^k})|}{2} q & q \text{ odd.} \end{cases}$$

This completes the proof of the second part of the theorem. \square

Remark 5.3. Suppose $m > 1$ and $m = 2^k l$ where k, l are integers with l odd. Then Theorem 5.2 can be stated more compactly as follows:

$$\begin{aligned} N_q(m, 2) &= \left(q - \frac{1 + (-1)^q}{2} \right) |V_m(1)| \\ &= \frac{1}{2} \left(q - \frac{1 + (-1)^q}{2} \right) \left(l |\mathcal{I}(l; q^{2^k})| - \left\lfloor \frac{1}{l} \right\rfloor \frac{1 + (-1)^{q-1}}{2} \right) \end{aligned}$$

where $\lfloor x \rfloor$ denotes the floor function. Note that

$$\left\lfloor \frac{1}{l} \right\rfloor \frac{1 + (-1)^{q-1}}{2} = \begin{cases} 1 & m \text{ is a power of 2 and } q \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.4. Suppose $m > 1$ and $m = 2^k l$ where k, l are nonnegative integers with l odd. Then

$$|TSRI(m, 2; q)| = \left(q - \frac{1 + (-1)^q}{2} \right) \left(\sum_{d|l} \mu(d) q^{\frac{m}{d}} - \left\lfloor \frac{1}{l} \right\rfloor \frac{1 + (-1)^{q-1}}{2} \right) \frac{|GL_m(\mathbb{F}_q)|}{2m(q^m - 1)}.$$

Proof. Follows from Theorem 4.2, Remark 5.3 and the fact that

$$|\mathcal{I}(l; q^{2^k})| = \frac{1}{l} \sum_{d|l} \mu(d) q^{\frac{2^k l}{d}}.$$

□

Theorem 5.5 (Carlitz). Let m be a positive integer and suppose $m = 2^k l$ for some integers k, l with l odd. The number of self-reciprocal irreducible monic (srir) polynomials of degree $2m$ in $\mathbb{F}_q[x]$ is equal to

$$\frac{1}{2m} \left(l |\mathcal{I}(l; q^{2^k})| - \left\lfloor \frac{1}{l} \right\rfloor \frac{1 + (-1)^{q-1}}{2} \right).$$

Proof. For $m = 1$ we need to count the number of b in \mathbb{F}_q such that $X^2 + bX + 1$ is irreducible in $\mathbb{F}_q[X]$. The polynomial $X^2 + bX + 1$ is irreducible precisely when b is not of the form $c + 1/c$ for some $c \in \mathbb{F}_q^*$. It is easily seen that

$$|\{c + 1/c : c \in \mathbb{F}_q^*\}| = \begin{cases} (q+1)/2 & q \text{ odd,} \\ q/2 & q \text{ even.} \end{cases}$$

The $m = 1$ case follows easily from this. Now suppose $m > 1$. Let the map $\Gamma : S_q(m, 2) \rightarrow \Delta_q(m, 2)$ be as in Theorem 4.2. Since all the fibers of Γ are of size m , it follows that the number of polynomials in $\Delta_q(m, 2)$ of the form $(1 + X)^m h\left(\frac{X^2}{1+X}\right)$ is equal to

$$\frac{1}{m} |\{X^2 - \lambda(X + 1) \text{ irreducible in } \mathbb{F}_{q^m}[X] : \mathbb{F}_{q^m} = \mathbb{F}_q(\lambda)\}| = \frac{|V_m(1)|}{m}.$$

Now

$$\begin{aligned} (1+X)^m h\left(\frac{X^2}{1+X}\right) \text{ is irreducible} &\Leftrightarrow X^m h\left(\frac{(X-1)^2}{X}\right) \text{ is irreducible} \\ &\Leftrightarrow X^m h_1\left(X + \frac{1}{X}\right) \text{ is irreducible} \end{aligned}$$

where $h_1(X) = h(X-2)$. Irreducible polynomials of the form $X^m h_1\left(X + \frac{1}{X}\right)$ where h_1 is monic of degree m are precisely the srin polynomials of degree $2m$. Thus the number of such polynomials is equal to $|V_m(1)|/m$. This is the statement of the corollary. \square

Remark 5.6. It is easy to show that any irreducible self-reciprocal polynomial of degree ≥ 2 over \mathbb{F}_q is necessarily of even degree.

Corollary 5.7. For every positive integer m , $f(X) \in \Delta_2(m, 2)$ if and only if $f(X-1)$ is a srin polynomial of degree $2m$.

Proof. This follows easily since polynomials in $\Delta_2(m, 2)$ are precisely the irreducible polynomials of the form

$$(1+X)^m h\left(\frac{X^2}{1+X}\right)$$

where h is monic of degree m . \square

Remark 5.8. If $f(X)$ is a self-reciprocal monic polynomial of degree $2m$ over \mathbb{F}_q , then $f(X+1)$ is $(m, 2)$ -decomposable. Further, if $f(X+1)$ is irreducible, then it is the characteristic polynomial of some matrix in $TSRI(m, 2; q)$. Such a matrix can easily be constructed from the $(m, 2)$ -decomposition of $f(X+1)$.

6. BOUNDS ON THE NUMBER OF TSRs

Theorem 6.1.

$$\begin{aligned} |TSRI(m, n; q)| &\leq \frac{|GL_m(\mathbb{F}_q)|}{q^m - 1} |\mathcal{J}(m; q)| q^{n-1}. \\ |TSRP(m, n; q)| &\leq \frac{|GL_m(\mathbb{F}_q)|}{q^m - 1} |\mathcal{P}(m; q)| q^{n-1}. \end{aligned}$$

Proof. First note that T is uniquely determined by $g_T(X)$ and $T_{(m)}$ (as in (2)). If $T \in TSRI(m, n; q)$, then $\phi_{T_{(m)}}(X)$ is irreducible of degree m and there are at most q^{n-1} possibilities for $g_T(X)$. The first bound easily follows from these observations. The second bound can be proved similarly by using Proposition 2.2. \square

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